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by Louis A. Kleiman and Raymond A. Arehart

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ABSTRACT

This paper describes a scheme to determine the size of the circular field of view that is both necessary and sufficient to include at least some specified number, n, of stars from a given set, independent of the orientation of the field within the celestial sphere. A geometrical proof of the scheme is presented, and all equations needed to effect the scheme are derived. The scheme is thus shown to be entirely analytical and to involve no assumptions concerning the distribution of the stars. Numerical results are presented in which the 1064 stars brighter than, or equal in brightness to, an apparent visual magnitude of +4.7 are considered. The size and location of the necessary fields of view are tabulated as a function of limiting star brightness for n=1, n=2, and n=3. Finally, the meaning and importance of the data are discussed and related to star sensor technology.

I.

INTRODUCTION

Future space missions of lengthened duration will impose increasingly demanding requirements on the reliability of space navigation systems. Strapdown inertial navigation and guidance technology, evolving from gimballed inertial system technology as a promising means of meeting these demands, has been reflected recently in a requirement for a gimballess star field sensor for three-axis attitude stabilization of the Apollo spacecraft (ref. 1).

Such a sensor must be able to detect either two stars in one field of view, or one star in each of two non-coaxial fields, to provide three-axis attitude error signals for the stabilization system. If it is assumed that the earth, the sun, the moon, and the planets are not seen by the star field sensor, and if the sensor is required to function at any time, independent of initial spacecraft attitude and orbital position, then proper operation can be absolutely assured only by designing the fields of view large enough to include at least the minimum necessary number (one or two) of stars brighter than the dimmest detectable star, for all possible orientations of the spacecraft.

The observation of at least three stars is required to determine spacecraft attitude by measurement of the angles subtended by the star pairs at the spacecraft. Thus a strapdown star-sensing instrument which can recognize any three-star pattern can be used to align a strapdown inertial navigation system with respect to the stars, and to measure the direction of the line of sight to a star with respect to the instrument. If, for simplicity of mechanical design, such an instrument employs a single field of view, the instrument can be made to operate independently of a spacecraft attitude control system by designing its field of view to include at least three stars for all possible orientations. The advantages of making star-angle measurements for navigation without having to point the spacecraft include not only a considerable saving of attitude control fuel and spacecraft energy, but also high reliability resulting both from the elimination of navigation system dependence upon the attitude control system, and from the relaxation of requirements on the attitude control system itself.

Strapdown star sensors with wide fields of view stand in the foreground of modern technology as potential contributors to highly reliable navigation, guidance, control, and stabilization systems of the future. This report describes an analytical scheme to determine precisely how wide the field of view must be to ensure the inclusion of at least a specified number, n, of stars from a given set, independent of the direction in which the field is pointing. Celestial bodies other than stars are assumed not to appear in the field of view at any orientation. Since most lenses look like disks from any point on the optical axis, the field of view considered here is circular. Once a catalog of star positions has been specified, the scheme yields not only the size of the field of view that is sufficient at all points, but also the location of the one point on the celestial sphere where that field is necessary for the observation of at least n stars. Since the size of the critical field of view is determined analytically, using the actual distribution rather than an assumed uniform distribution of stars, there is no need to add a "margin of safety," which might impose excessive requirements on optical design, or otherwise degrade performance of the star sensor.

II.

THE SCHEME

The scheme desribed in this section determines the size of the circular field of view that is both necessary and sufficient for the observation of a minimum of n stars, independent of the orientation of the field at the center of the celestial sphere. Any circular field of view can be represented by a right circular cone, the vertex of which is the center of the celestial sphere, the center-line or axis of which lies along the pointing direction of the field, and the base of which is bounded by a small circle on the surface of the sphere (see figure 1).

For the purposes of this discussion, the field of view is considered to be the surface area of the spherical segment defined by the base of the cone, that is, the area on the sphere within the small circle boundary. The center of the field, then, is the point of intersection of the surface of the sphere and the axis of the cone.

The size of the circular field of view is expressed throughout the report in terms of the angle θ between the axis of the cone and any straight line on the surface of the cone, as shown in figure 2. This "angular radius" is just half the "angular diameter" or "total field angle," 2θ, subtended at the center of the sphere by lines to two points at the ends of any diameter of the base of the cone.

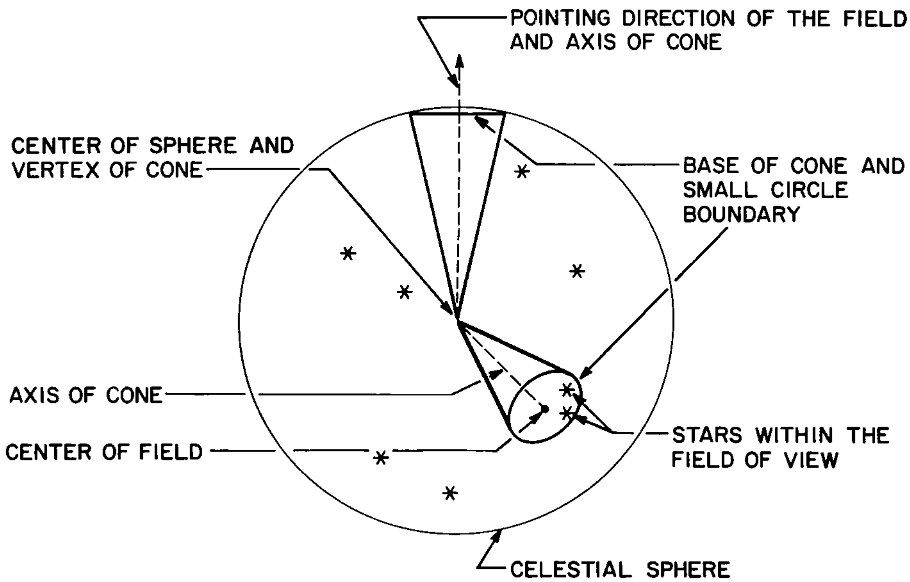


Figure 1. - Conical Representation of Fields of View on the Celestial Sphere

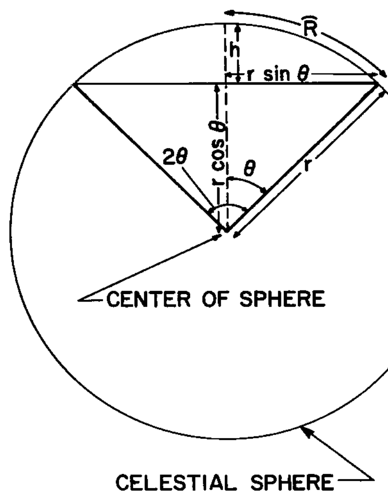


Figure 2. - Geometry of a Field of View

Since the length of a great circle arc $\hat{R}$ which subtends the angle θ at the center of a sphere of radius r is given by

$$\hat{R} = r\theta, \tag{1}$$

where θ is expressed in radians, the "arc radius," $\hat{R}$, can also be used to indicate the size of the field of view (see figure 2). In fact, for a celestial sphere of unit radius, the number of units of length in the arc radius, $\hat{R}$, of any field of view is equal to the number of radians in its angular radius, θ (ref. 2). In general, the concept of an angular radius, θ, or angular diameter, 2θ, is a simple means of representing the size of a circular field of view. However, it may be more convenient to consider the arc radius, $\hat{R}$, of the field when its small circle boundary is swung with a compass on a celestial globe in the laboratory, since the angle between the compass arms does not equal the angular radius, θ, of the field of view unless the length of the arms equals the radius of the globe.

The angular radius, θ, of the field of view is here related to a solid angle, Ω, which is equal in units of "steradians" to the area of the spherical segment defined by the field on a sphere of unit radius. The formula for the area of a spherical segment is

$$A = 2\pi rh \tag{2}$$

where r is the radius of the sphere, and h is the height of the segment. For a sphere of unit radius, the geometry of figure 2 leads to the following equation:

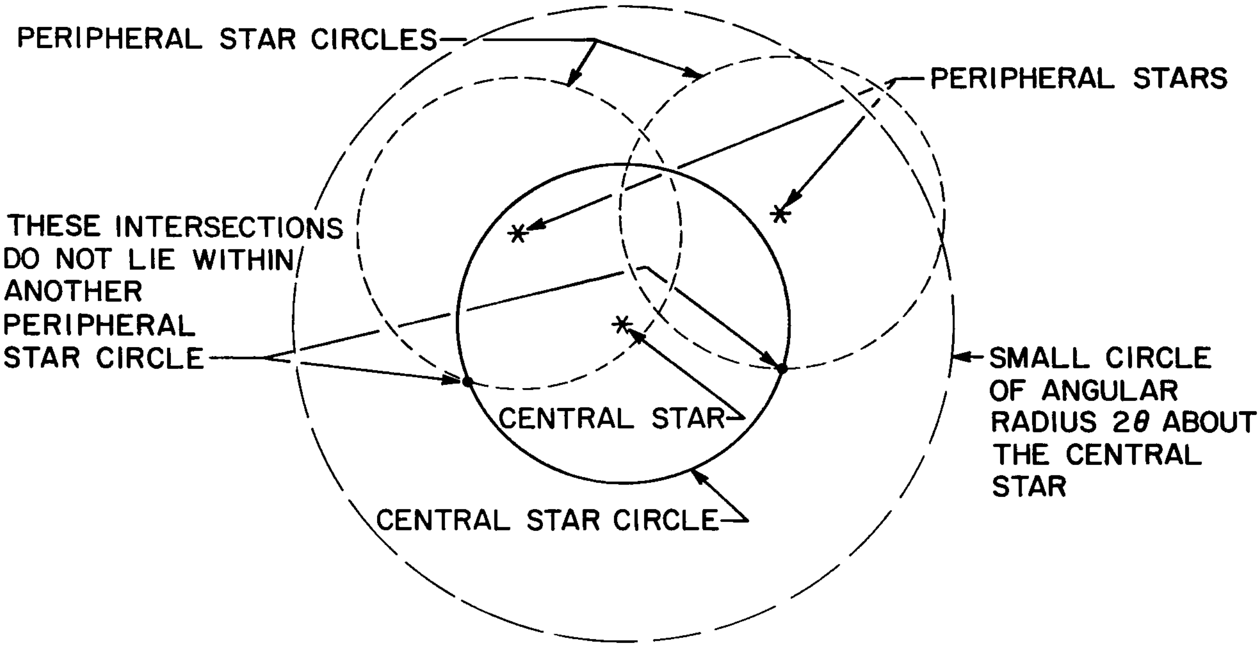
$$\Omega\,(\text{steradians}) = 2\pi(1 - \cos\theta). \tag{3}$$

Thus, a hemispherical field of view ($\theta = \frac{\pi}{2}$), for instance, contains 2π steradians.

The scheme requires that the apparent positions of a set of stars, all stars brighter than apparent visual magnitude +3.0, for example, be given. Throughout the report, star positions and the directions of points on the celestial sphere are specified by the common astronomical parameters called right ascension, α, and declination, δ, analogous to longitude and latitude, or to azimuth and elevation. Each star of the set is considered independently as a "central star." After choosing arbitrarily an angular radius, θ, for the field of view, perhaps 30 degrees, all the stars within 2θ of the central star are tabulated. Such stars are referred to hereafter as "peripheral stars" of a given central star.

The scheme proceeds by requiring that every point on the celestial sphere which is an angle θ away from both the central star and any of its peripheral stars lie within θ of a second peripheral star. As noted in figure 3 then, it can be seen that each of the intersections of a small circle with angular radius θ about the central star (central star circle) and small circles with angular radii θ about the peripheral stars (peripheral star circles) must lie within a second peripheral star circle. If this requirement is met for every star in the set considered independently as a central star, then the observation of at least one star ($n = 1$) at all orientations is assured for the chosen angular radius θ. Failure to meet the requirement necessitates increasing the field of view and repeating the test. Since any central star which satisfies the imposed necessary criterion for a given θ also satisfies it for an angular radius larger than θ, any field of view larger than or equal to the necessary, or critical, field of view is sufficient for the observation of at least one star. Determination of the necessary and sufficient field, then, involves an iterative process, incrementing θ when the test is failed and decrementing θ when the test is passed, in order to converge on the critical value of θ.

For the case $n>1$, an arbitrary θ is again chosen, and the same criterion must be satisfied for each central star as when $n = 1$. An additional requirement is that every point on the celestial sphere which is an angle θ away from any two peripheral stars, and which lies within θ of the central star, must also be within θ of $n-1$ other peripheral stars. As shown in figure 4 for $n = 2$, this criterion means that every intersection of any two peripheral star circles which lies inside the central star circle must also lie within $n-1$ other peripheral star circles. If this requirement is met for every star in the set considered independently as a central star, then the field of view is assured of including at least n stars for any orientation. The size of the necessary and sufficient field of view is obtained, as when $n = 1$, by incrementing and decrementing θ appropriately to converge on the critical angular radius.



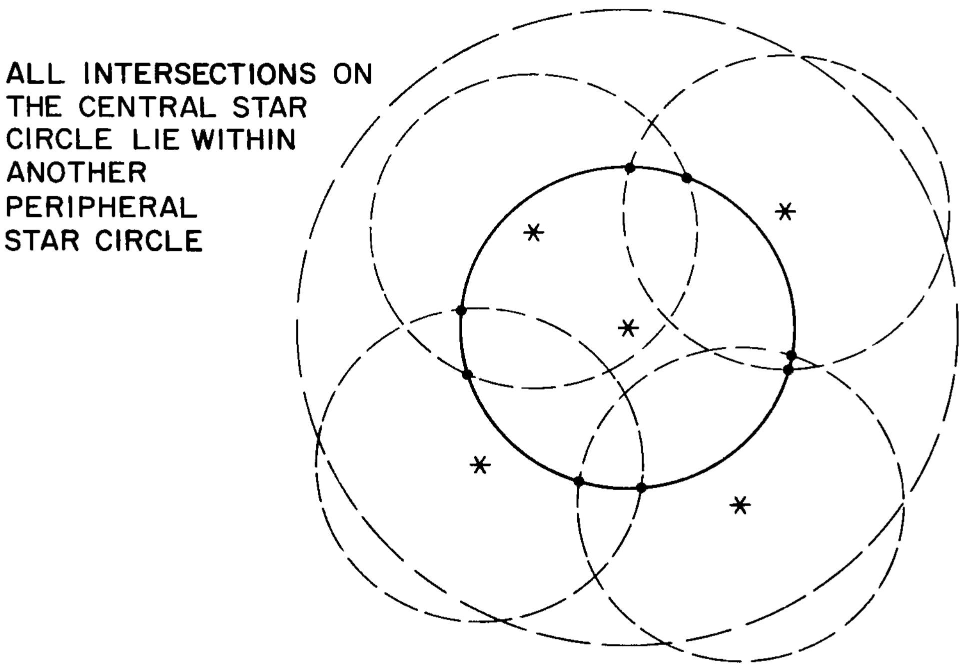


Figure 3. - Geometry of Central Star and Peripheral Star Circles for n=1

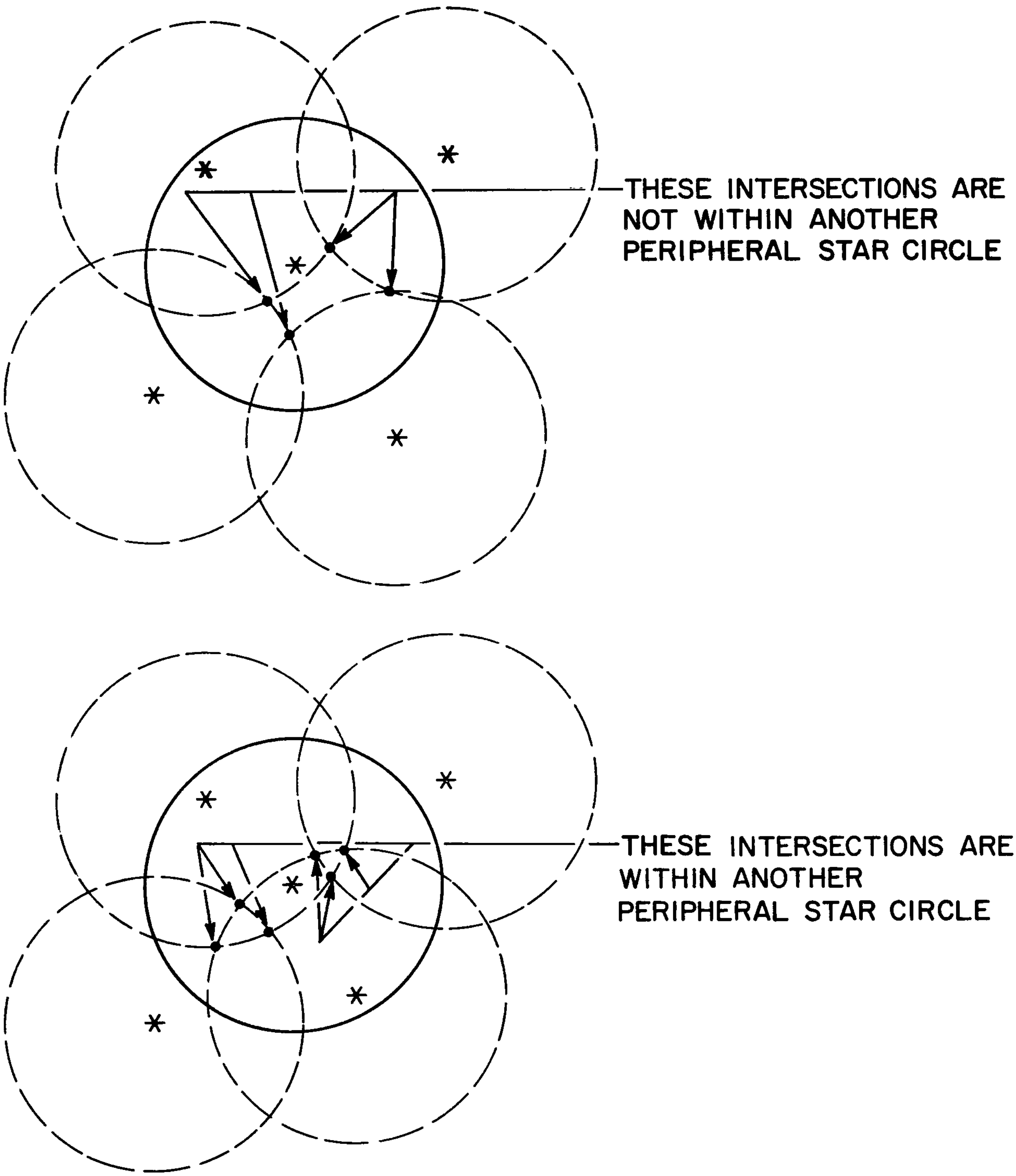


Figure 4. - Geometry of Central Star and Peripheral Star Circles for n=2

III.

PROOF OF THE SCHEME

The validity of the scheme is proven here for n=2. Following the proof is a discussion of the geometry of the problem which justifies the scheme for an arbitrary n. The proof is presented in two parts; the conclusion of the first theorem, coupled with the hypothesis of the second, forms the statement of the scheme.

The first theorem is that two stars are included in a circular field of view, independent of the direction in which it is pointing, if, and only if, each central star circle is "covered" by its peripheral star circles. The term "covered" is used to describe the total concealment of the central star circle which would result if the interior of the peripheral star circles were opaque, as shown in figure 5. Thus, if a central star circle is covered, each of its points and the points of its interior are defined to be within θ of at least one peripheral star.

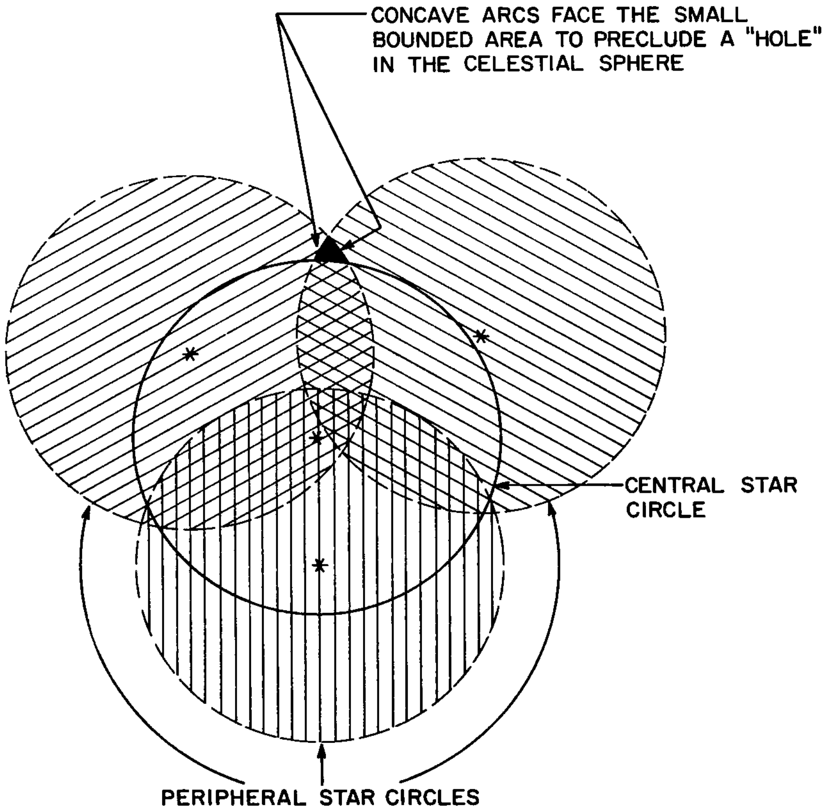


Figure 5. - Coverage of a Central Star Circle

Proof of the necessity of the hypothesis is as follows. Suppose there exists a central star circle which is not covered by its peripheral star circles. Then a field of view pointed at the uncovered area includes only one star, the central star. Thus, the hypothesis must be a necessary condition for the validity of the conclusion.

Sufficiency of the hypothesis is proven by contradiction. Assume that each central star circle is covered, but that fewer than two stars are included in the field of view at various points on the celestial sphere. The boundaries of all the sets of such points are arcs of a number of central star circles. Similarly, the boundaries of the sets of points where at least two stars are included are arcs of at least two central star circles. Since the central star circles have been assumed to be covered, no boundary of a set of points where fewer than two stars are included can be a circle. Otherwise, central star circles would have to be tangent to the circular boundary at every point. A finite number of stars, and hence of central star circles, precludes this possibility. Thus, the boundaries of all sets of points where fewer than two stars are included in the field of view are arcs of two or more central star circles. But any point on these boundaries is a point of a central star circle which is not covered. Since the last statement contradicts the hypothesis, there is evidently no point on the celestial sphere at which the field of view can be directed such that fewer than two stars appear in the field, if the hypothesis is true. Then the hypothesis is a sufficient condition for the validity of the conclusion.

The second theorem is that a central star circle is covered by its peripheral circles if, and only if, each intersection of any two peripheral star circles that lies within θ of the central star, and each intersection of a central star circle and a peripheral star circle lie within θ of another peripheral star.

The necessity of the hypothesis of the second theorem is proven by supposing that one intersection of each of the two types stated in the hypothesis is not within θ of another peripheral star. Then the points in the immediate neighborhood of each of the points of intersection are members of four distinct sets, one of which is the set of points within θ of only the central star. If the field of view is directed at a point of such a set, it contains only the central star. Since, to be covered, every point of a central star circle and of its interior must, by definition, be within θ of at least one peripheral star, the hypothesis must be necessary for the validity of the conclusion.

Contradiction is used again to prove the sufficiency of the hypothesis of the second theorem. Assume the truth of the hypothesis, but suppose that there are points in the central star circles which are not covered. By the same argument used to prove sufficiency in the first theorem, the boundary of the set of all such points is a combination of two or more arcs of peripheral star circles. Then the intersections of these peripheral star circles are not within θ of another peripheral star. Thus, the hypothesis is contradicted and, since every point of the central star circle is covered, the hypothesis is sufficient for the validity of the conclusion.

An intuitive feel for the scheme can be gained by considering small circles of an arbitrary angular radius θ about each of the stars on the celestial sphere. Sufficiently large circles result in a total coverage of the sphere. As θ is decreased, however, "holes" in the cover are opened, the simplest of which is an area bounded by arcs of three small circles, as shown in figure 6. The convex "side" of the arcs faces the hole, thus indicating that any point within the hole is not within θ of any of the three stars around which the small circles are centered.

The holes begin to open, as θ is decreased, at the intersections of the small circles about the stars. The interior of any one small circle is the locus of points at which a field of view can be pointed to assure the inclusion of at least one star. But in "stepping" from one small circle to the next, every intersection of two small circles must be within a third small circle to avoid stepping into a hole. If θ is slightly larger than necessary to preclude stepping into a hole, two concave arcs face the area bounded by arcs of each of the three small circles in question (see figure 5).

When it is desired to observe n stars in the field of view at all orientations, the celestial sphere must be covered with n thicknesses of small circles about the stars. Thus, as n is increased for a given set of stars, θ must be increased to fill the holes in each layer.

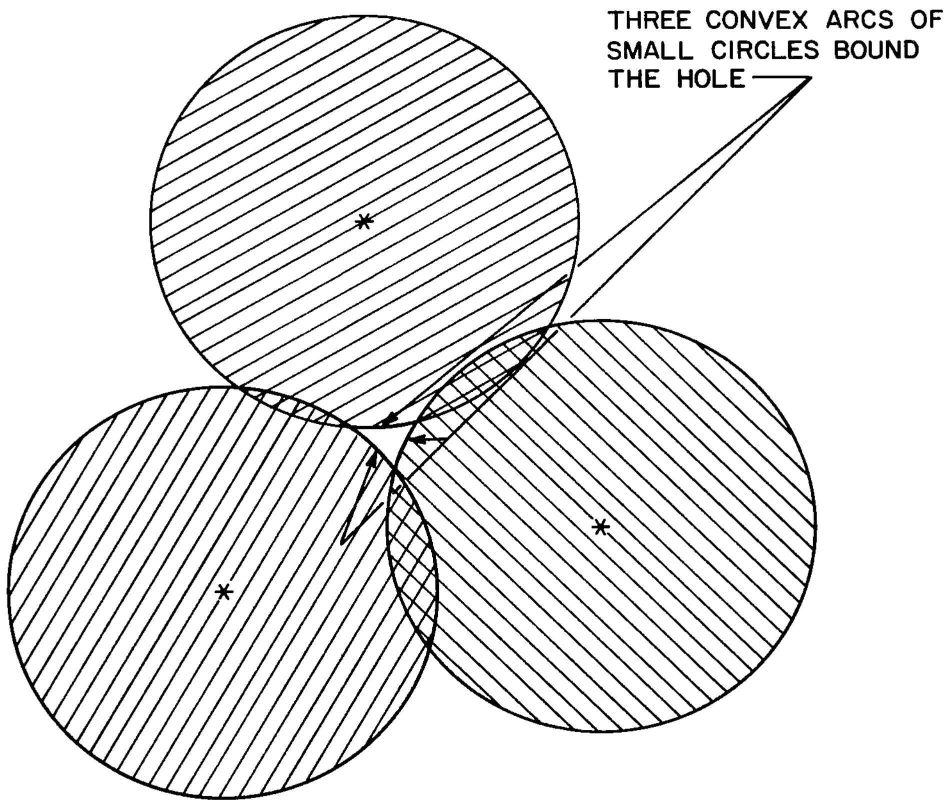


Figure 6. - A Hole in the Celestial Sphere

IV.

DERIVATION OF EQUATIONS

As described in Sections II and III, the scheme makes extensive use of intersections of small circles on a sphere. In this section, equations are derived for the right ascension and declination of the two points of intersection of two small circles with angular radii θ, given the right ascension and declination of the center of each of the small circles on a sphere of unit radius. The geometry of the problem is shown in figure 7.

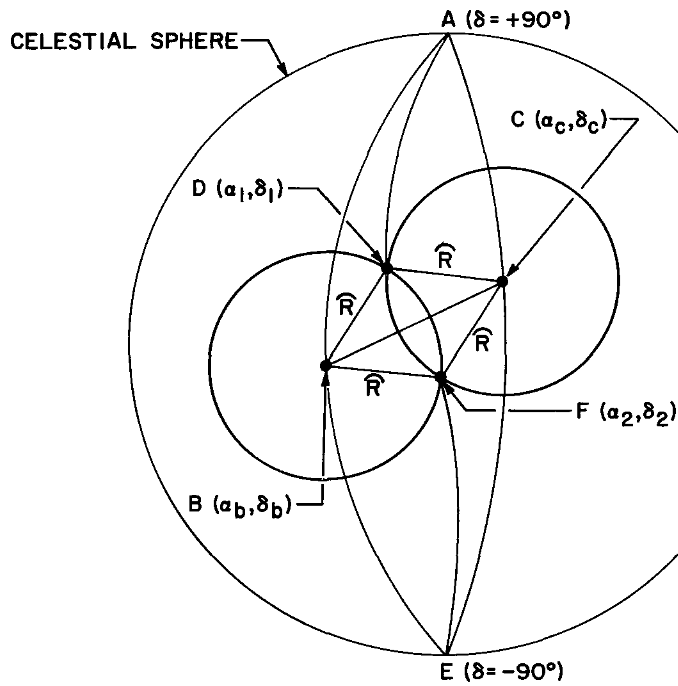


Figure 7. - Geometry for Derivation of Equations for Intersections of Two Small Circles on the Celestial Sphere

The known points are A (δ = +90°), B (α_b, δ_b), C (α_c, δ_c), and E (δ = -90°). The angular radius, θ, and arc radius $\hat{R}$, of the small circles are assumed to be known. Thus, $\hat{R}$ is equal to the great circle arcs $\widehat{BD}$, $\widehat{DC}$, $\widehat{CF}$, and $\widehat{FB}$. $\widehat{AB}$, $\widehat{AD}$, $\widehat{AC}$, $\widehat{EB}$, $\widehat{EF}$, $\widehat{EC}$, and $\widehat{BC}$ are also great circle arcs. The points D (α_1, δ_1) and F (α_2, δ_2) are to be found.

Since a sphere of unit radius is assumed, all great circle arcs are hereafter described by the angle which they subtend at the center of the sphere. Using this concept of circular or angular measure for great circle arcs, the following relations can be determined directly from figure 7:

$$\overset{\frown}{AB} = 90° - \delta_b \tag{4}$$

$$\overset{\frown}{AC} = 90° - \delta_c \tag{5}$$

$$\overset{\frown}{AD} = 90° - \delta_1 \, . \tag{6}$$

Since great circles passing through the celestial poles are meridians, the angle between two meridian planes defines the spherical angles BAC, BAD, and DAC as

$$BAC = \alpha_c - \alpha_b \tag{7}$$

$$BAD = \alpha_1 - \alpha_b \tag{8}$$

$$DAC = \alpha_c - \alpha_1 . \tag{9}$$

From the law of cosines for spherical trigonometry:

$$\cos \overset{\frown}{BD} = \cos \overset{\frown}{AB} \cos \overset{\frown}{AD} + \sin \overset{\frown}{AB} \sin \overset{\frown}{AD} \cos BAD \tag{10}$$

$$\cos \overset{\frown}{DC} = \cos \overset{\frown}{AD} \cos \overset{\frown}{AC} + \sin \overset{\frown}{AD} \sin \overset{\frown}{AC} \cos DAC \tag{11}$$

$$\cos \overset{\frown}{BC} = \cos \overset{\frown}{AB} \cos \overset{\frown}{AC} + \sin \overset{\frown}{AB} \sin \overset{\frown}{AC} \cos BAC . \tag{12}$$

Substituting (4) through (9) into (10) through (12) results in

$$\cos \theta = \sin \delta_1 \sin \delta_b + \cos \delta_1 \cos \delta_b \cos (\alpha_1 - \alpha_b) \tag{13}$$

$$\cos \theta = \sin \delta_1 \sin \delta_c + \cos \delta_1 \cos \delta_c \cos (\alpha_1 - \alpha_c) \tag{14}$$

$$\cos \overset{\frown}{BC} = \sin \delta_b \sin \delta_c + \cos \delta_b \cos \delta_c \cos (\alpha_c - \alpha_b) . \tag{15}$$

If (13) and (14) could be solved for α_1 and δ_1 at this point, the task would be simple. The presence of trigometric functions of the unknowns, however, necessitates the aid of further spherical trigonometry to obtain the solution.

Again from the law of cosines:

$$\sin \delta_1 = \cos \overset{\frown}{AD} = \cos \theta \sin \delta_b + \sin \theta \cos \delta_b \cos ABD . \tag{16}$$

But:

$$\cos ABD = \cos (ABC - DBC)$$

$$= \cos ABC \cos DBC + \sin ABC \sin DBC \tag{17}$$

Defining:

$$C_{bc} = \cos \widehat{BC}$$

$$= \sin \delta_b \sin \delta_c + \cos \delta_b \cos \delta_c \cos (\alpha_c - \alpha_b) \tag{18}$$

and applying the law of cosines to spherical triangle DBC,

$$\cos DBC = \frac{\cos \theta}{\sin \theta} \sqrt{\frac{1 - C_{bc}}{1 + C_{bc}}} . \tag{19}$$

Since, from the geometry of the problem, $0° < DBC < 90°$,

$$\sin DBC = + \sqrt{1 - \cos^2 DBC} . \tag{20}$$

The law of sines applied to spherical triangle ABC gives:

$$\frac{\sin ABC}{\sin \widehat{AC}} = \frac{\sin BAC}{\sin \widehat{BC}} \tag{21}$$

or

$$\sin ABC = \frac{\cos \delta_c \sin (\alpha_c - \alpha_b)}{\sqrt{1 - C^2_{bc}}} , \tag{22}$$

while the law of cosines yields

$$\cos ABC = \frac{\sin \delta_c - C_{bc} \sin \delta_b}{\cos \delta_b \sqrt{1 - C^2_{bc}}} . \tag{23}$$

Finally, substituting (17) through (20), (22), and (23) back into (16) yields:

$$\sin \delta_1 = \frac{\cos \theta (\sin \delta_b + \sin \delta_c) + \cos \delta_b \cos \delta_c \sin (\alpha_c - \alpha_b) \sqrt{\frac{C_{bc} - \cos 2\theta}{1 - C_{bc}}}}{1 + C_{bc}} . \tag{24}$$

The computer algorithms used to obtain numerical results require both the sine and cosine of α_1 and δ_1. Thus, since the range of δ_1 is:

$$-90° < \delta_1 < +90°, \tag{25}$$

it follows that

$$\cos \delta_1 = + \sqrt{1 - \sin^2 \delta_1} . \tag{26}$$

Sin α_1 is found by solving (13) for cos α_1 and substituting the result into (14) which can then be solved for sin α_1 to yield:

$$\sin \alpha_1 = \frac{\begin{array}{c}\cos \theta (\cos \alpha_b \cos \delta_b - \cos \alpha_c \cos \delta_c) \\ + \sin \delta_1 (\sin \delta_b \cos \delta_c \cos \alpha_c - \cos \delta_b \sin \delta_c \cos \alpha_b)\end{array}}{\sin (\alpha_c - \alpha_b) \cos \delta_1 \cos \delta_b \cos \delta_c} . \tag{27}$$

Cos α_1, from (13), is then:

$$\cos \alpha_1 = \frac{1}{\cos \alpha_b} \left[\frac{\cos \theta - \sin \delta_1 \sin \delta_b}{\cos \delta_1 \cos \delta_b} - \sin \alpha_b \sin \alpha_1 \right] . \tag{28}$$

Note the symmetry of (27) with respect to the alphabetic subscripts. Interchanging "b" and "c" merely negates both the numerator and denominator, leaving the result of the equation for sin α_1 unchanged. The computer algorithms employed, therefore, are independent of the order in which the stars are selected from the catalog. Similarly, (28) yields the same result if the subscript "b" is replaced by "c" throughout. An alternate solution is thus available if a singularity results in (28) when cos α_b is zero.

The equation for sin δ_2 is found in a manner similar to that for sin δ_1 by considering relations in the spherical triangles BEC, BFC, BEF, and FEC. The final result is:

$$\sin \delta_2 = \frac{\cos \theta (\sin \delta_b + \sin \delta_c) - \cos \delta_b \cos \delta_c \sin (\alpha_c - \alpha_b) \sqrt{\dfrac{C_{bc} - \cos 2\theta}{1 - C_{bc}}}}{1 + C_{bc}} . \tag{29}$$

Cos δ_2, sin α_2, and cos α_2 are identical in form to (26) through (28) with "1" replaced by "2" throughout.

If α_b equals α_c, a zero appears in the denominator of (27). In this special case, (24) and (29) still yield the correct values for sin δ_1 and sin δ_2. In fact, δ_1 is identically equal to δ_2 and is given by

$$\delta_1 = \delta_2 = \frac{\delta_b + \delta_c}{2} . \tag{30}$$

The appropriate functions of α_1 and α_2 can then be found from the following equations, which arise from the simplified geometry of the special case:

$$\alpha_1 = \alpha_b + \cos^{-1}\left[\frac{\cos\theta - \sin\delta_1 \sin\delta_b}{\cos\delta_1 \cos\delta_b}\right] \tag{31}$$

$$\alpha_2 = \alpha_b - \cos^{-1}\left[\frac{\cos\theta - \sin\delta_1 \sin\delta_b}{\cos\delta_1 \cos\delta_b}\right]. \tag{32}$$

V.

A SIMPLER SCHEME

The most undesirable part of the original scheme, particularly in the execution of computer programs, is the iterative procedure to find the critical value of θ. Nevertheless, the technique works and was, in fact, used a number of times to obtain data which was cross-checked on a celestial globe with a compass.

One output of the original computer program was the identification of the last intersection at which the tests for a sufficient θ failed as θ was increased. This output made possible the location of the last hole in the celestial sphere to be covered, and thus the one point on the celestial sphere where the critical field of view was indeed necessary. A compass set to span the critical arc radius and swung about the last point of failure passed directly through the central and peripheral stars, or two peripheral stars, as predicted, but it also appeared to pass directly through a third star! Further study of this phenomenon led to the formulation of the following simple scheme.

The field of view necessary and sufficient for the observation of n stars at all orientations is the largest small circle, determined by any three stars on the celestial sphere, within which there are exactly n-1 stars. If three stars are present on the boundary of a field of view, at least one of them must appear in the field if θ is increased infinitesimally. That one star on the boundary, plus n-1 stars within, assures the observation of n stars for the field and the pointing direction in question. Thus, by choosing the largest "gap" in the celestial sphere, that is, the largest small circle that surrounds only n-1 stars, the necessary and sufficient field of view is automatically determined.

The particular advantage of the simpler scheme is the elimination of the need for an iterative procedure to find the critical θ. The analytical expression for θ is derived as follows with the aid of figure 8.

From Section IV, equation (18), it is known that

$$C_{bc} = \cos \overset{\frown}{BC}$$

$$= \sin \delta_b \sin \delta_c + \cos \delta_b \cos \delta_c \cos (\alpha_c - \alpha_b) \qquad (33)$$

$$C_{cd} = \cos \overset{\frown}{CD}$$

$$= \sin \delta_c \sin \delta_d + \cos \delta_c \cos \delta_d \cos (\alpha_d - \alpha_c) \tag{34}$$

$$C_{bd} = \cos \overset{\frown}{BD}$$

$$= \sin \delta_d \sin \delta_b + \cos \delta_d \cos \delta_b \cos (\alpha_b - \alpha_d). \tag{35}$$

Applying the law of cosines to the three small spherical triangles in figure 8 yields:

$$\cos BEC = \frac{C_{bc} - \cos^2 \theta}{1 - \cos^2 \theta} \tag{36}$$

$$\cos CED = \frac{C_{cd} - \cos^2 \theta}{1 - \cos^2 \theta} \tag{37}$$

$$\cos BED = \frac{C_{bd} - \cos^2 \theta}{1 - \cos^2 \theta}. \tag{38}$$

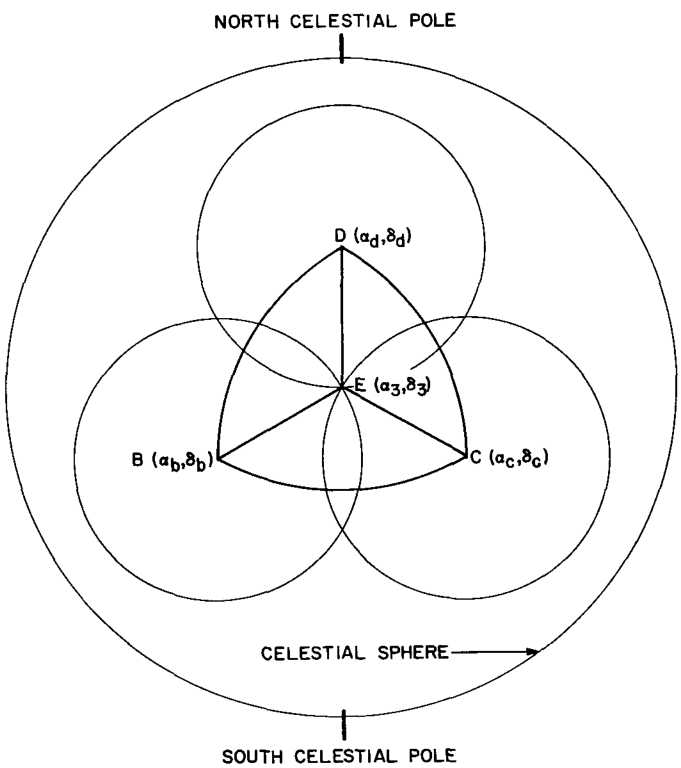


Figure 8. - Geometry for Derivation of Equations Involving the Common Intersection of Three Small Circles on the Celestial Sphere

But:

$$\cos \text{BED} = \cos (360° - \text{CED} - \text{BEC})$$

$$= \cos (\text{CED} + \text{BEC})$$

$$= \cos \text{CED} \cos \text{BEC} - \sin \text{CED} \sin \text{BEC}, \tag{39}$$

and from the geometry of the problem:

$$\sin \text{CED} = + \sqrt{1 - \cos^2 \text{CED}} \tag{40}$$

$$\sin \text{BEC} = + \sqrt{1 - \cos^2 \text{BEC}}. \tag{41}$$

Substituting (36) through (38), (40), and (41) into (39), after considerable algebra, results in

$$\cos \theta = \sqrt{\frac{C_{bc}^2 + C_{cd}^2 + C_{bd}^2 - 2C_{bc} C_{cd} C_{bd} - 1}{C_{bc}^2 + C_{cd}^2 + C_{bd}^2 + 2 (C_{bc} + C_{cd} + C_{bd}) - 2 (C_{bc} C_{cd} + C_{bc} C_{bd} + C_{cd} C_{bd}) - 3}}. \tag{42}$$

The coordinates of point $E(\alpha_3, \delta_3)$, the point at which a centered field of view of the critical angular radius is necessary for the observation of n stars, are found from equations developed in Section IV. The three small circles in figure 8 are considered in pairs. Equations (24) and (29) are evaluated for each star pair, and the value which is the same for each pair is the proper $\sin \delta_3$. $\cos \delta_3$, $\sin \alpha_3$, and $\cos \alpha_3$ are given by (26) through (28), with numerical subscripts changed appropriately.

VI.

NUMERICAL RESULTS

The basic set of stars used to obtain the numerical results presented in this section is that set listed under "Mean Places of the Stars, 1967.0," in reference 3. Since the purpose of the scheme is to ensure absolutely the observation of n stars, all stars listed by reference 3 as variable in magnitude are excluded from the basic set if they become dimmer than apparent visual magnitude +4.7. Similarly, the largest value of apparent visual magnitude for variable stars which appears in references 4, 5, or 6 is the value which is considered. A total of 1064 stars, 11 of which are variable, results. The apparent visual magnitude of each variable star at its dimmest is listed in Table I.

TABLE I

MAXIMUM VALUE OF APPARENT VISUAL MAGNITUDE FOR ELEVEN VARIABLE STARS

No.	Star	Apparent Magnitude Visual, m_v
1.	γ Cas	+3.0
2.	ρ Per	+4.1
3.	β Per	+3.5
4.	ϵ Aur	+3.8
5.	α Ori	+1.2
6.	η Gem	+4.2
7.	β Lyr	+4.3
8.	R Lyr	+4.5
9.	η Aql	+4.4
10.	μ Cep	+4.7
11.	δ Cep	+4.4

The total field angles, 2θ, of the critical fields of view are plotted in figure 9 versus limiting apparent visual magnitude, m_V, for n=1, n=2, and n=3. The discrete data points are listed in table II, as well as the number of stars involved with each calculation. Increments of 0.1 magnitude, consistent with reference 3, determine the appropriate set of stars for each calculation. Figure 10 shows the number of stars brighter than or equal to a given value of apparent visual magnitude, m_v.

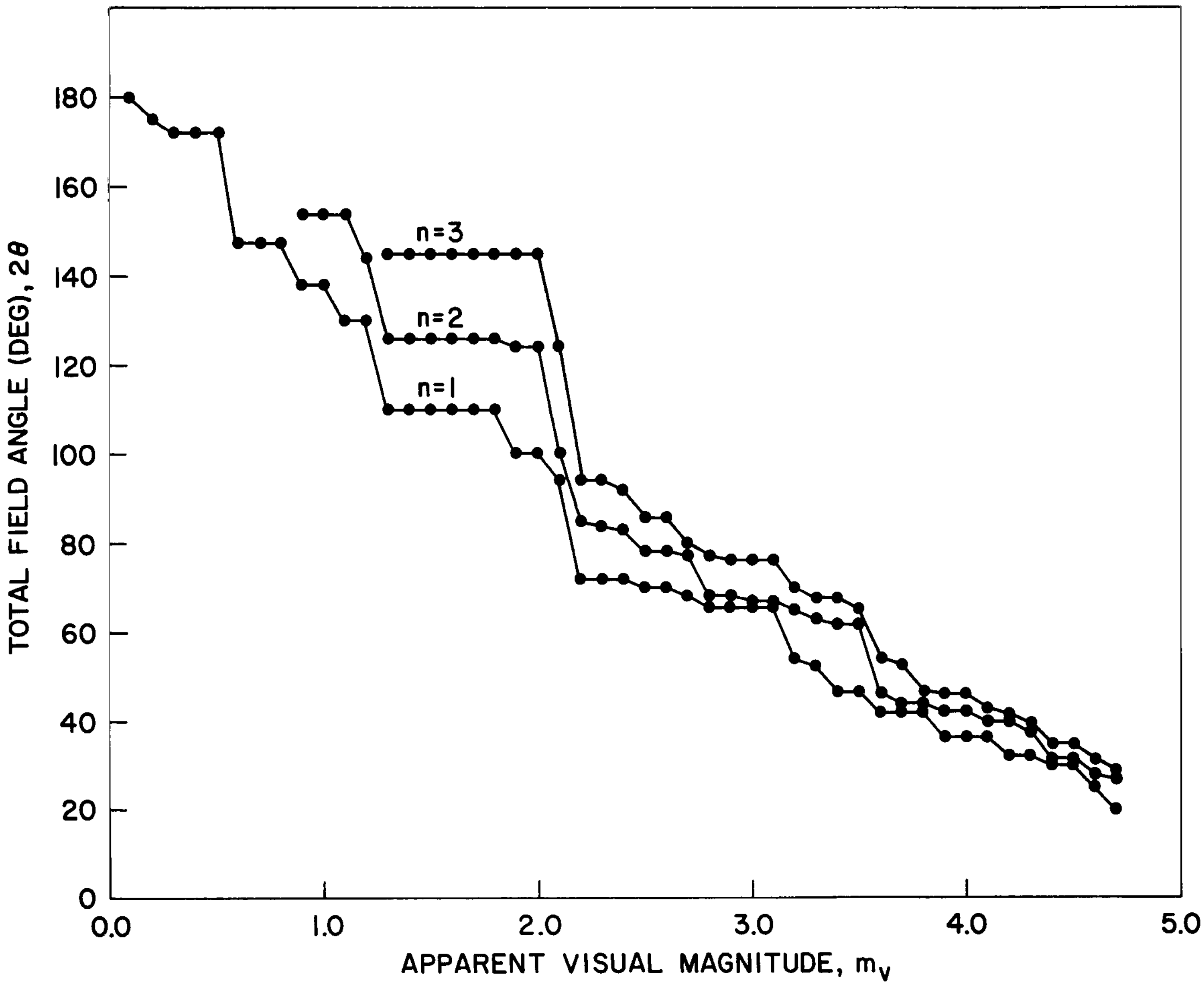


Figure 9. - Critical Field of View versus Limiting Magnitude, m_v

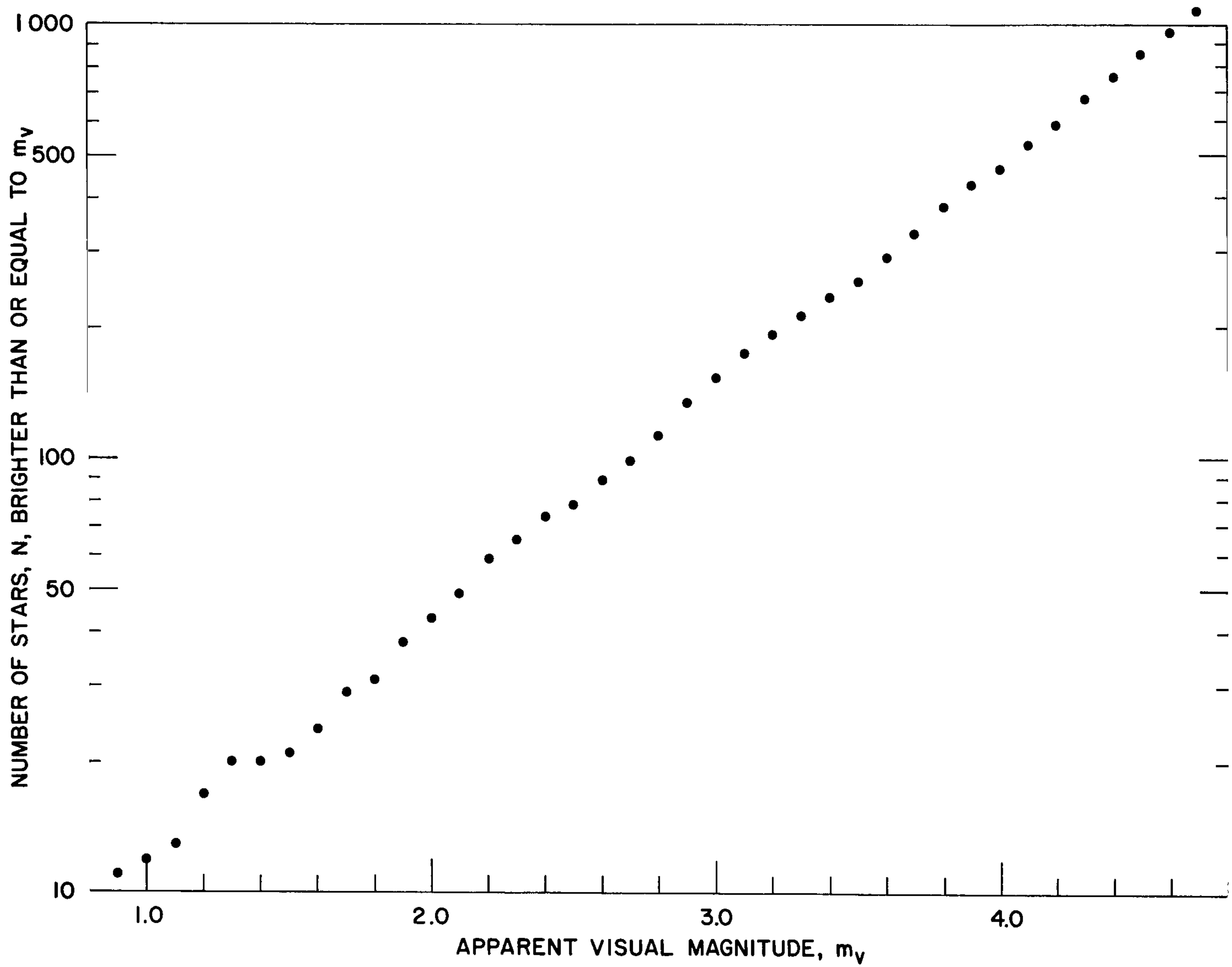


Figure 10. - Star Population versus Limiting Magnitude, m_v

TABLE II

ANGULAR RADII, θ, OF CRITICAL FIELDS OF VIEW

m_v	N	$\theta°$, n=1	$\theta°$, n=2	$\theta°$, n=3	m_v	N	$\theta°$, n=1	$\theta°$, n=2	$\theta°$, n=3
0.1	4	89.784	>90	>90	2.5	79	34.780	39.140	43.052
0.2	6	87.647	>90	>90	2.6	89	34.780	39.140	43.052
0.3	7	86.177	>90	>90	2.7	99	34.217	38.627	40.292
0.4	7	86.177	>90	>90	2.8	112	32.710	34.217	38.627
0.5	8	86.177	>90	>90	2.9	134	32.710	34.217	38.411
0.6	9	73.698	>90	>90	3.0	153	32.710	33.577	38.411
0.7	9	73.698	>90	>90	3.1	175	32.710	33.577	38.411
0.8	9	73.698	>90	>90	3.2	193	26.951	32.710	34.817
0.9	11	69.239	77.329	>90	3.3	213	26.035	31.555	33.666
1.0	12	69.239	77.329	>90	3.4	236	23.457	31.292	33.666
1.1	13	65.114	77.329	>90	3.5	255	23.457	31.292	32.466
1.2	17	65.114	72.543	>90	3.6	291	21.148	23.309	27.315
1.3	20	55.358	63.228	72.543	3.7	330	21.148	22.162	26.434
1.4	20	55.358	63.228	72.543	3.8	378	21.148	22.096	23.309
1.5	21	55.358	63.228	72.543	3.9	427	18.280	21.006	22.972
1.6	24	55.358	63.228	72.543	4.0	465	18.280	21.006	22.972
1.7	29	55.358	63.228	72.543	4.1	526	17.660	19.767	21.362
1.8	31	55.358	63.228	72.543	4.2	587	16.140	19.767	20.471
1.9	38	50.484	62.426	72.543	4.3	675	16.140	18.653	19.493
2.0	43	50.484	62.426	72.543	4.4	754	15.053	15.624	17.264
2.1	49	46.805	50.484	62.426	4.5	852	15.053	15.624	17.001
2.2	59	36.044	42.571	46.805	4.6	962	12.492	14.027	15.522
2.3	65	36.044	42.048	46.805	4.7	1064	10.269	13.521	14.423
2.4	74	36.044	41.530	45.806					

TABLE III

CENTERS OF CRITICAL FIELDS OF VIEW

m_v	n=1		n=2		n=3	
	α°	δ°	α°	δ°	α°	δ°
0.1	12.622	4.900	-	-	-	-
0.2	359.833	- 7.485	-	-	-	-
0.3-0.5	349.796	-16.581	-	-	-	-
0.6-0.8	358.491	13.187	-	-	-	-
0.9-1.0	7.650	10.497	192.702	15.787	-	-
1.1	2.892	5.388	192.702	15.787	-	-
1.2	2.892	5.388	3.752	-23.718	-	-
1.3-1.8	10.124	19.489	5.273	3.958	3.752	-23.718
1.9-2.0	1.405	17.803	2.233	2.483	3.752	-23.718
2.1	30.043	-10.821	1.405	17.803	2.233	2.483
2.2	333.385	4.792	315.377	2.846	30.043	-10.821
2.3	333.385	4.792	340.955	12.151	30.043	-10.821
2.4	333.385	4.792	339.426	11.502	325.943	- 1.576
2.5-2.6	47.110	-26.674	38.835	-15.137	31.062	- 5.516
2.7	46.693	-24.500	39.514	-16.349	38.073	-18.466
2.8	46.585	-28.719	46.693	-24.500	39.514	-16.349
2.9	46.585	-28.719	46.693	-24.500	39.054	-16.257

m_v	n=1		n=2		n=3	
	α°	δ°	α°	δ°	α°	δ°
3.0-3.1	46.585	-28.719	45.190	-29.616	39.054	-16.257
3.2	37.703	-32.114	46.585	-28.719	22.116	-22.617
3.3	52.422	-38.954	18.160	-12.695	12.533	- 9.286
3.4	35.200	-18.310	18.260	-12.338	12.533	- 9.286
3.5	35.200	-18.310	18.260	-12.338	16.309	-11.247
3.6	79.448	67.109	105.887	66.486	154.276	-15.062
3.7	79.448	67.109	163.753	- 6.483	151.056	-14.298
3.8	79.448	67.109	88.465	67.449	105.887	66.486
3.9-4.0	72.655	71.074	79.638	68.006	86.071	81.115
4.1	91.170	71.925	192.762	32.589	91.214	66.283
4.2	100.110	73.575	192.762	32.589	193.390	31.544
4.3	100.110	73.575	107.232	78.765	83.926	65.833
4.4	111.484	74.837	104.507	74.158	102.909	76.009
4.5	111.484	74.837	104.507	74.158	8.844	-25.627
4.6	105.037	70.946	79.922	74.312	14.463	-28.490
4.7	72.240	-84.360	15.140	-31.013	14.166	-29.577

Some very interesting conclusions can be drawn from figure 9. The flat portion of each graph, from apparent visual magnitude 1.3 through 1.8 for n=1 and n=2, and from 1.3 through 2.0 for n=3, indicates that an increase of sensitivity in this particular range gains nothing as far as reducing the necessary field of view is concerned. If, however, the detector of a star sensor is made sensitive to stars of magnitude 2.2, the required radius of the field of view drops sharply. For the case n=1, it is interesting to note that the total angle of the critical field of view decreases from approximately 30° to 25° to 20° as m_v drops in successive steps from 4.5 to 4.6 to 4.7.

The centers of the critical fields of view are listed in table III and plotted in figure 11 (Mercator Projection) and in figures 12 and 13 (North and South celestial polar projections). The location of the center of a critical field of view is the pointing direction on the celestial sphere where stars of interest are least likely to appear. Thus, the distribution of these points indicates areas of low star density.

The North Galactic Pole, which lies, according to the new (1959) system of galactic coordinates, in the direction $\alpha = 12^h49^m = 192.25°$ and $\delta = +27.4°$ for the equinox 1950.0, and the South Galactic Pole, $\alpha = 49^m = 12.25°$ and $\delta = -27.4°$, appear in figures 11, 12, and 13. These poles provide a convenient reference for the division of critical pointing directions into three groups, as shown in the figures. Group I consists of all points within 60° of the North Galactic Pole; Group II consists of all points within 65° of the South Galactic Pole; and Group III consists of all points between 60° and 75° from the North Galactic Pole. The members of each group are classified in Table IV according to n and m_v. The existence of Groups I and II supports the common claim that the stars are least dense near the galactic poles. Group III, however, points out that an area within 30° of the Galactic Equator has fewer stars of apparent visual magnitude 3.6 to 4.6 than have the galactic polar regions. Future mission plans involving star sensors, then, should take into account the scarcity of certain classes of stars in the Group III region near the Galactic Equator.

TABLE IV

CLASSIFICATION OF CRITICAL POINTING DIRECTIONS

Group I		Group II		Group III	
n=1	none	n=1	m_v = 0.1-3.5; 4.7	n=1	m_v = 3.6-4.6
n=2	m_v = 0.9-1.1; 3.7; 4.1-4.2	n=2	m_v = 1.2-3.5; 4.7	n=2	m_v = 3.6; 3.8-4.0; 4.3-4.6
n=3	m_v = 3.6-3.7; 4.2	n=3	m_v = 1.3-3.5; 4.5-4.7	n=3	m_v = 3.8-4.1; 4.3-4.4

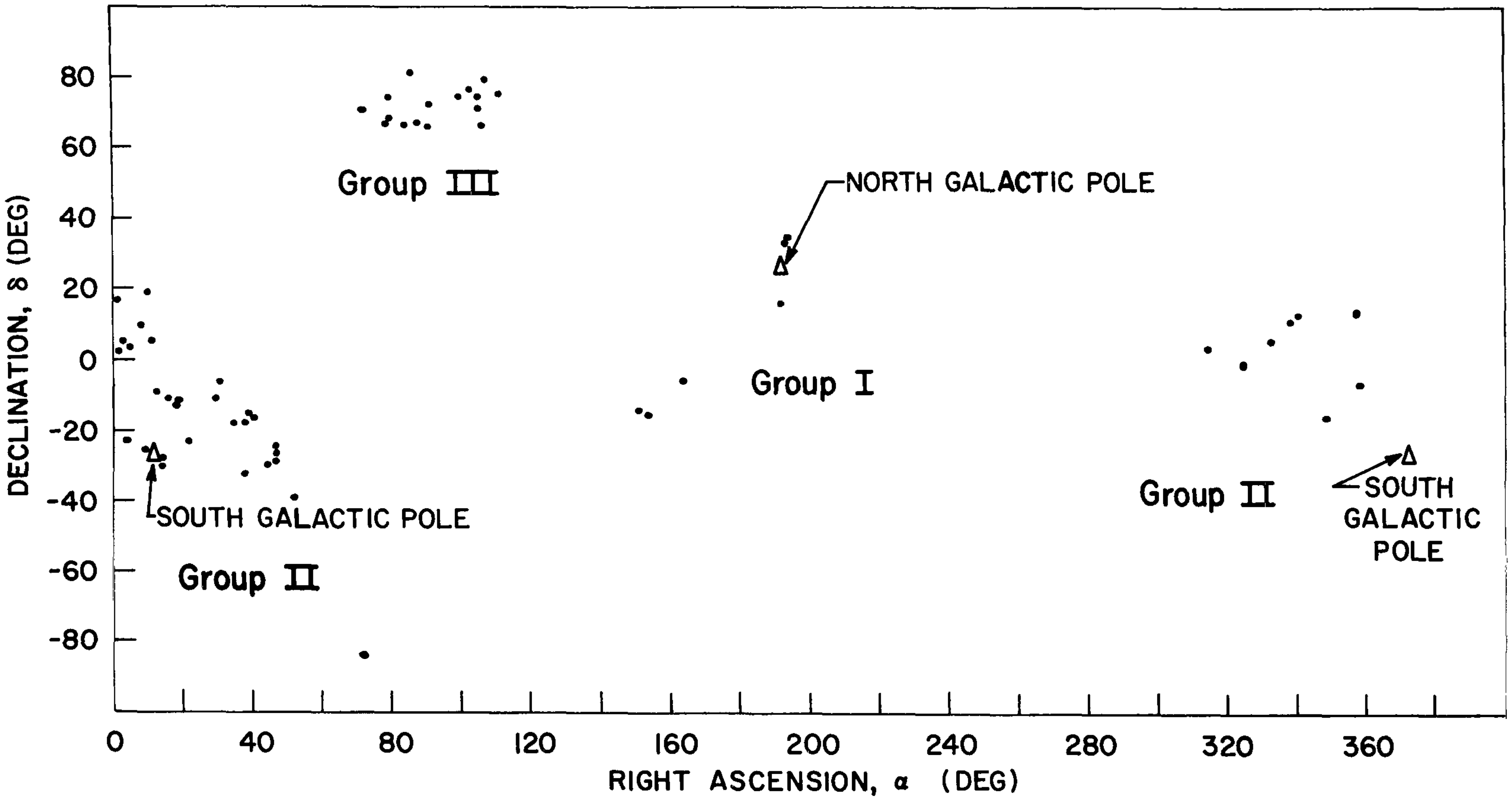


Figure 11. - Mercator Projection of Critical Pointing Directions

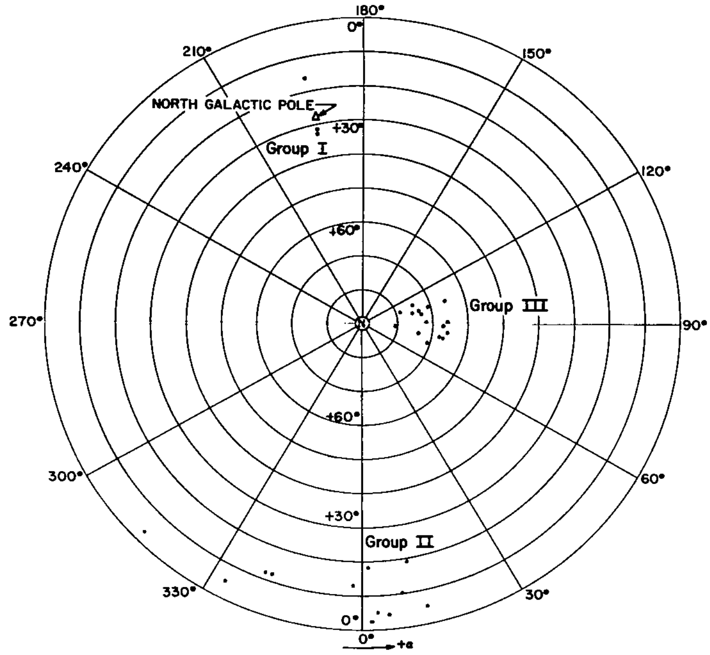


Figure 12. – North Celestial Polar Projection of Critical Pointing Directions

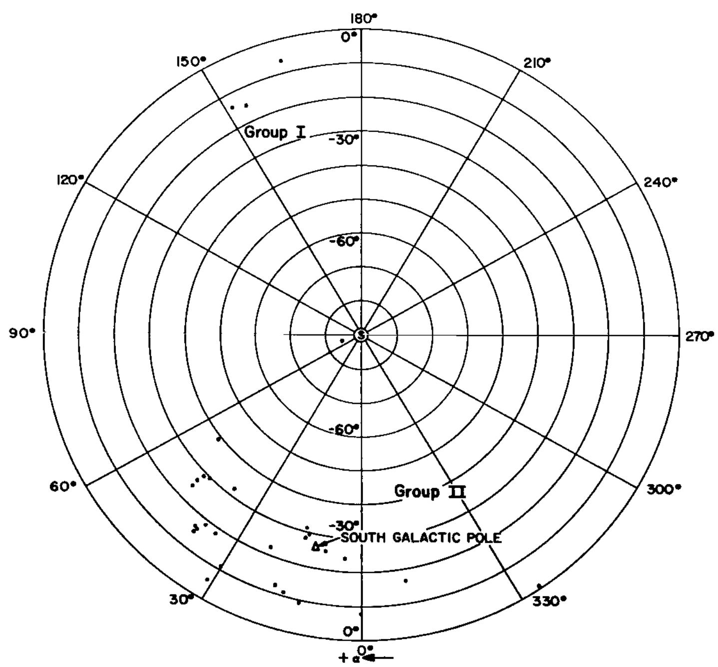


Figure 13. – South Celestial Polar Projection of Critical Pointing Direction

The algorithm used to obtain numerical results involves three assumptions which reduce the otherwise excessive computation time resulting from the many combinations of three stars from the star sets. There are more than 2×10^8 combinations of three stars, for example, from the set of 1064 stars brighter than or equal to magnitude 4.7. Without the simplifying assumptions, each of the 2×10^8 combinations would take at least 1 msec of computer time, or a total of more than 55 hours per point on figure 9.

The first assumption is that the star set which results from each increment of apparent visual magnitude causes the new value of θ to be less than or equal to the preceding value, since the addition of the dimmer stars to the star set can only "fill" the holes in the celestial sphere. Next, no two stars in combination with a third star can form a small circle of angular radius less than θ if the great circle arc joining the two stars subtends an angle greater than 2θ. Finally, if no new star, added to a preceding star set by a magnitude increment, is within the preceding critical field of view, then the critical field for the new star set has the same angular radius θ. Otherwise, the new θ is smaller, since a hole is filled. These three assumptions allow many combinations and computations to be skipped.

VII.

CONCLUSIONS

The operation of strapdown star sensors at any orientation of the spacecraft on which they are mounted necessitates a field of view which is large enough to include the required number of stars when pointing at the most barren spot on the celestial sphere. The scheme described herein enables the size of the necessary field to be determined analytically, as well as the center of the most barren spot. By eliminating statistical techniques and assumed star distributions from the process of determination of a sufficient field of view, one eliminates also the possibility of failure of the statistical scheme to specify a large enough field. Thus, there can be no possibility of failure of a star sensor because of an inadequate field of view if the analytical technique is employed during the design of the sensor. In a reliability analysis of the sensor, then, the probability of occurrence of this particular failure mode is zero.

The numerical results show that if reliable strapdown star sensors are to become a reality, they must have detectors sensitive to fairly dim stars and fields of view significantly wider than most statistical studies to date have indicated.

VIII.

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125-17-02-11

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